SLE: an introduction

Chordal SLE.

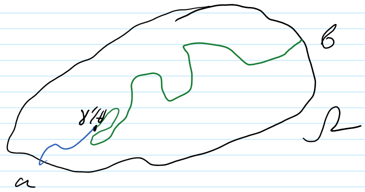
Motivation: Want a random curve $\gamma = \gamma_{(\Omega,a,b)}$ in simply connected Ω, joining two prime ends $a, b \in \partial\Omega$, satisfying the following.

1) Conformal invariance:

$\varphi: \Omega_1 \to \Omega_2$ conformal, $\varphi(a_1)=a_2$, $\varphi(b_1)=b_2$ $\Rightarrow$ then $\varphi(\gamma(\Omega_1,a_1,b_1)) = \gamma(\Omega_2,a_2,b_2)$ in law

2) Markov property:

$$\gamma_{(\Omega,a,b)} \mid \gamma' \overset{\text{in law}}{=} \gamma_{(\Omega\setminus\gamma',\, \gamma'(t),\, b)} \cup \gamma'$$



Want: γ to be Löwner:

Ω_t = Component of $\Omega \setminus \gamma(0,t]$ containing b at the boundary.

then (Ω_t) should form Löwner chain (chordal).

$(t \to \Omega_t$ – continuous, $\Omega_t \subsetneq \Omega_s$ $t<s)$

Enough to consider only one domain (say $\mathbb{H}$) and define there, others by conformal invariance.

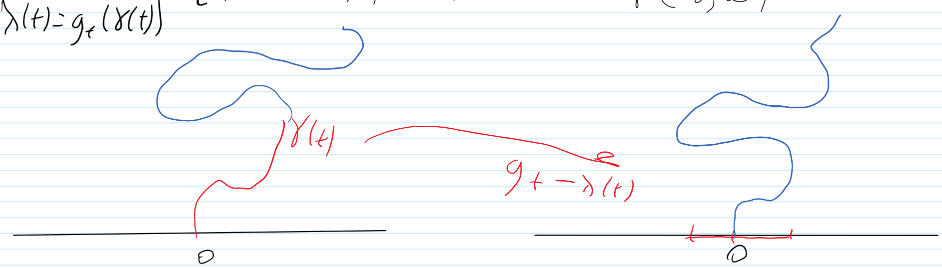
Then 1) + 2) becomes

2'): conformal Markov property:

$g_t: \mathbb{H}\setminus\gamma(0,t] \to \mathbb{H}$ with hydrodynamic normalization.

Then $g_t(\gamma[t,\infty)) - \lambda(t) \overset{\text{in law}}{=} \gamma(0,\infty)$

$\lambda(t) = g_t(\gamma(t))$



Thm (Schramm) Let γ be a random Löwner curve which satisfies 1) + 2'). Then its driving function is $B(\kappa t)$ for some κ if it has finite expectation $(E|\lambda(t)|<\infty)$ $\forall t$.

Pf. Let $\lambda(t)$ be the driving function of γ. Notation: $G_t := g_t - \lambda(t)$.

Know $G_{t+s} \mid G_t \overset{\text{in law}}{=} G_t \circ G_s$.

Expand G_{t+s} at ∞:

$$z - \lambda(t+s)\Big|_{G_t} = (z - \lambda(t) + \dots) \circ (z - \lambda(s) + \dots) =$$

$$z - (\lambda(t) + \lambda(s)) + \dots$$

so $(\lambda(t+s) - \lambda(t))\Big|_{G_t} \overset{\text{law}}{=} \lambda(s)$. so $\lambda(t+s) - \lambda(t)$ is stationary

(independent on $\lambda(t)$ or t)'. Also $E(\lambda(t+\delta)) - E(\lambda(t)) = E(\lambda(\delta))$'
so $E(\lambda(t)) = \alpha t$ for some α.
Also $E(\lambda(t+\delta) - E(\lambda(t+\delta)) \mid G_t) = E(\lambda(t+\delta) - \lambda(t) + \lambda(t) - E(\lambda(t+\delta)) \mid G_t)$
$E(\lambda(\delta)) + \lambda(t) - E(\lambda(t+\delta)) = \lambda(t) - E(\lambda(t))$.
Thus $\lambda(t) - \alpha t = B(\kappa t)$, by Levy Thm.

Now, rescaling by c ($z \to cz$) is a conformal map of $\mathbb{H}$, does not change the law.
So $c^{-1}\lambda(c^2 t) \overset{law}{=} \lambda(t)$. Thus $\alpha = 0$ ∎

Remark: The opposite is also true. If $\lambda(t) = B(\kappa t)$, the corresponding process in $\mathbb{H}$ satisfies 2').

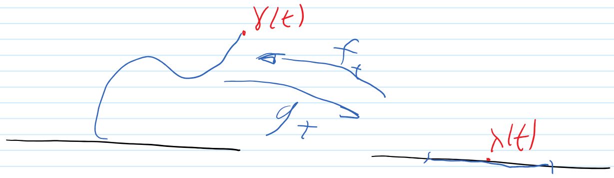
Def. SLE_κ in $(\mathbb{H}, 0, \infty)$ is the Löwner evolution with the driving force $B(\kappa t) \overset{d}{=} \sqrt{\kappa}\, B(t)$.

Remark SLE_κ satisfies
1) Scaling: $\tilde{g}_t(z) = c^{-1} g_{c^2 t}(cz) - SLE_\kappa$.
2) Strong Markov: $\tilde{g}_t(z) = g_{\tau+t} \circ g_\tau^{-1}(z + B(\kappa\tau)) - B(\kappa\tau)$
SLE_κ $\forall$ stopping time τ.
For $\tilde{g}_t$, the driving function is $\tilde{\lambda}(t) = B(\kappa(t+\tau)) - B(\kappa\tau) \overset{d}{=} B(\kappa t)$.

Def. SLE_κ in (Ω, a, b) (Ω - simply connected domain, $a, b \in \partial\hat{\Omega}$ - prime ends)
$\varphi: (\mathbb{H}, 0, \infty) \to (\Omega, a, b)$ - conformal.
SLE_κ in $\Omega := \varphi(SLE_\kappa$ in $\mathbb{H})$. Does not depend on the choice of φ (by scaling invariance).

Theorem (Rohde - Schramm) SLE_κ in $\mathbb{H}$ is generated by a Löwner curve a.s.

Proof ($\kappa \neq 8$)



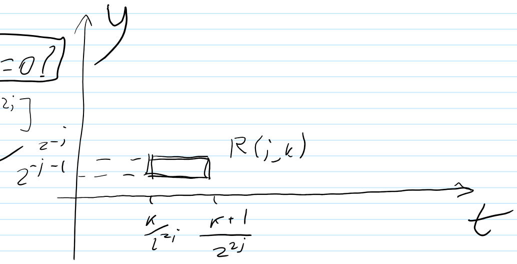
Notation: $\hat{f}_t(z) := f_t(z + B(\kappa t))$ ($\hat{f}_t(0) = \gamma(t)$, if exists).
As we proved, only need to show that $\forall t\ \exists \lim\limits_{y \to 0+} \hat{f}_t(iy)$, continuous in t.
Enough to fix T, and prove for $0 \le t \le T$.
By scaling, can take $T = 1$.

Notation continued: $H(y,t) := \hat{f}_t(iy)$. $0 \le t \le 1$ $y > 0$ $\boxed{y = 0?}$ $\uparrow y$

Notation continued: $H(y,t) := \hat f_t(iy)$. $0 \le t \le 1$ $y > 0$ $\boxed{y=0?}$

$j,k \in \mathbb{N}$, $k < 2^{2j}$. $R(j,k) := [2^{-j-1}, 2^{-j}] \times [k2^{-2j}, (k+1)2^{-2j}]$;

$d(j,k) := \operatorname{diam} H(R(j,k))$.



Claim $\exists\, \sigma > 0$ $\sum_{j=0}^{\infty} \sum_{k=0}^{2^{2j}-1} P(d(j,k) \ge 2^{-j\sigma}) < \infty$

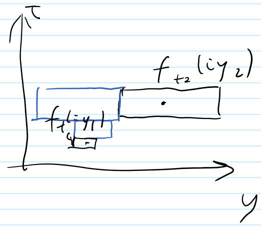
$(\le 2^{-(2+\varepsilon)j})$ – later

By Claim + Borel-Cantelli: for all but finitely many $j,k \in \mathbb{N}$: $d(j,k) \le 2^{-j\sigma}$. So $d(j,k) \le C2^{-j\sigma}$ for some random C.

So $|\hat f_{t_1}(iy_1) - \hat f_{t_2}(iy_2)| \le \sum_{j} (d(j,k^1_j) + d(j,k^2_j)) \le A\, 2^{-j\sigma}$ $\quad |H(y_1,t_1) - H(y_2,t_2)|$

$j = \min(-\log_2 y_1, -\log_2 y_2, -\frac{1}{2}\log_2 |t_1 - t_2|)$

(go to the common ancestor).



So when $(y,t) \to (0,t_0) \Rightarrow \hat f_t(iy) \to \hat f_{t_0}(iy)$ – converges.

$\hat f_t(0)$ is continuous in t!

Claim follows from the following key estimate (we'll prove later!)

Theorem (Rohde–Schramm). Let $b \in [0, 1 + \frac{4}{\kappa}]$,

$a = 2b + \frac{\kappa b(1-b)}{2}$, $\lambda = 4b + \kappa b(1-2b)/2$.

Then $\exists\, c(\kappa,b)$: $\forall t \in [0,1]$, $y > 0$, $\delta \in [0,1]$, $x \in \mathbb{R}$.

$$P\left(|\hat f_t'(x+iy)| > \frac{\delta}{y}\right) \le c(\kappa,b)\left(1 + \frac{x^2}{y^2}\right)^b \left(\frac{y}{\delta}\right)^\lambda A(\delta, a-\lambda)$$

where

$$A(\delta,\mu) := \begin{cases} \delta^{-\mu}, & \mu > 0 \\ 1 + |\log \delta|, & \mu = 0 \\ 1, & \mu < 0 \end{cases}$$

Apply to $b = \frac{8+\kappa}{4\kappa}$, then $\lambda = \frac{(\kappa+8)^2}{16\kappa}$, $a = \frac{3\kappa^2 + 32\kappa + 64}{32\kappa}$.

We get $P\left(|\hat f_s'(i2^{-j})| > 2^j \frac{2^{-j\sigma}}{j^2}\right) \le C\, 2^{-2j} 2^{-\varepsilon j}$ for some $\varepsilon > 0$

if $\quad \sigma < \dfrac{(\kappa-8)^2}{\max\left((\kappa+8)^2, \frac{1}{2}(3\kappa^2 + 32\kappa + 64)\right)}$ $\quad (*)$

This gives us control of $|\hat f_t'(i2^{-j})|$ – distortion of diameter!

Does not work for $\kappa = 8$!

Idea: subdivide $[k2^{-2j}, (k+1)2^{-2j}]$ into subintervals with controlled change of $B(\kappa t)$.

Define: $t_0 := (k+1)2^{-2j}$, $t_{n+1} := \sup\{t < t_n : |B(kt) - B(kt_n)| = 2^{-j}\}$.
$N := \min\{n : t_n \le t_0 - 2^{-2j} = k2^{-2j}\}$
$t_\infty := t_0 - 2^{-2j} = k2^{-2j}$, $\hat{t}_n := \max(t_n, t_\infty)$.

By scaling of BM, $\exists \rho$: $P(N > 1) = \rho$ (independent on j or k)
By Markov, $P(N \ge m+1 / N \ge m) \le \rho$, so
$P(N \ge m) \le \rho^m$.

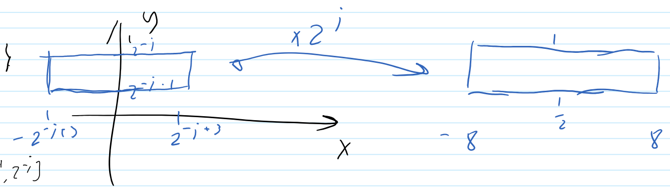
Observe: $\hat{f}_s'$ is measurable with respect to σ-field generated by $\{B(kt), t \le s\}$. $\hat{t}_n$ - generated by $B(kt)$, $t \ge \hat{t}_n$.

So, $\forall s \in [t_\infty, t_0]$, $\delta > 0$, we have:
$P(|\hat{f}_{t_n}'(i2^{-j})| > \delta \mid \hat{t}_n = s\} = P(|\hat{f}_s'(i2^{-j})| > \delta)$. - independence!

So $P(|\hat{f}_{t_n}'(i2^{-j})| > \delta \mid \hat{t}_n > t_\infty) \le \sup_{s \in [0,1]} P(|\hat{f}_s'(i2^{-j})| > \delta)$.

So $P(\exists n : |\hat{f}_{t_n}'(i2^{-j})| > \delta) \le$
$P(|\hat{f}_{t_\infty}'(i2^{-j})| > \delta) + \sum_{n=0} P(|\hat{f}_{\hat{t}_n}'(i2^{-j})| > \delta \mid \hat{t}_n > t_\infty\}$
$\le E(N+1) \sup_{s \in [0,1]} P(|\hat{f}_s'(i2^{-j})| > \delta) \le C \sup_{s \in [0,1]} P(|\hat{f}_s'(i2^{-j})| > \delta)$

Let $S := \{x + iy : |x| \le 2^{-j+3},\ y \in [2^{-j-1}, 2^{-j+3}]\}$



Claim $(R(j,k)) \subset \bigcup_{n \le N} \hat{f}_{t_n}(S)$.
$f_{\hat{t}_n}(S) \cap f_{\hat{t}_{n+1}}(S) \ne \emptyset$

Pf of Claim: Let $t \in [\hat{t}_{n+1}, \hat{t}_n]$, $y \in [2^{-j-1}, 2^{-j}]$
Then $\hat{f}_t(iy) = \hat{f}_{\hat{t}_{n+1}}\big(\overbrace{g_{\hat{t}_{n+1}}(f_t(iy)) - B(k\hat{t}_{n+1})}^{I}\big)$

Check: $I \in S$. Taking $t = \hat{t}_n$ - implies the second statement.
Let $\varphi(s) := g_s(\hat{f}_t(iy))$, $s \le t$; then $\varphi(t) = iy + B(kt)$.
$\partial_s \varphi(s) = \frac{2}{\varphi(s) - B(ks)}$, $\partial_s \operatorname{Im} \varphi_s < 0$, so $\operatorname{Im} \varphi(s) \ge \operatorname{Im} \varphi(t) \ge 2^{-j-1}$
so $|\partial_s \varphi_s| \le 2^{j+2}$. So, since $|t - \hat{t}_{n+1}| \le 2^{-2j}$, we have
$|\varphi(\hat{t}_{n+1}) - \varphi(t)| \le 2^{2-j}$. Since $|B(kt) - B(k\hat{t}_{n+1})| \le 2^{1-j}$,
So $I \in S$. Since if $(y, t) \in R(j,k)$, then $y \in [2^{-j-1}, 2^{-j}]$
$t \in [\hat{t}_{n+1}, \hat{t}_n]$ for some n, the Claim follows ■

But $\exists a : \frac{|\hat{f}_t'(z)|}{|\hat{f}_t'(i2^{-j})|} \le a$ for $z \in S$ (Bloch estimate, since hyperbolic diameter of S bounded independently of j)
So $\operatorname{diam} \hat{f}_t(S) \le C 2^{-j} |\hat{f}_t'(i2^{-j})|$.
So $d(j,k) \le \sum_{n=0}^{N} \operatorname{diam}(\hat{f}_{t_n}(S)) \lesssim 2^{-j} \sum_{n=0}^{N} |\hat{f}_{t_n}'(i2^{-j})| \le$
$C 2^{-j} N \max\{|\hat{f}_{t_n}'(i2^{-j})|, n \le N\}$.

(*)

So $P(d(j,k) \ge 2^{-j\sigma}) \le P(N > j^2) + j^2 P(|\hat{f}_{t_n}'(i2^{-j})| > 2^j \frac{2^{-j\sigma}}{j^2}) \le$
$\rho^{j^2} + C 2^{-(2+\varepsilon)j} \le C_1 2^{-(2+\varepsilon)j}$ ■